Chapter

APPLICATIONS OF STOCHASTIC HYBRID SYSTEMS IN PORTFOLIO OPTIMIZATION

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ABSTRACT

Stochastic Optimal Control has an important role in financial mathematics and is extensively used in various applications. We present applications of Stochastic Hybrid models in finance to account for regime switching dynamics. Stochastic models with a motion part and additionally a jump part are able to capture abrupt fluctuations that are usual phenomenon in financial markets. These kinds of models allow for more realistic investigation of portfolio optimization and utility maximization in financial markets.

The Chapter starts with an illustration of the Efficient Market Hypothesis in the theory of finance and a description of jump models governed by a Markov process. On this basis, Stochastic Hybrid models with insider information are implemented in portfolio optimization models. The conditions under which insider investors have larger information sets are incorporated into the models by using filtration techniques as a necessary tool. The models comprise portfolio optimization with optimal investment and consumption strategies. Explicit consideration of risk aversion in an optimal investment and consumption problem allows for optimality conditions that are related to specific risk types in a market.

A more general model for portfolio optimization is established afterwards. The Chapter ends with a presentation of a model for insurance and reinsurance with insider information in a utility maximization framework.

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1. INTRODUCTION

Efficient market hypothesis (EMH) asserts that financial markets are "informationally efficient", and exceptions are explained by arbitrage phenomenon. According to the efficient market hypothesis it is impossible to consistently achieve returns in excess of market returns on risk adjusted basis, because it is assumed that relevant market information is already incorporated in the market prices of a traded asset. Market efficiency suggests that competitive prices must reflect price changes that perform a random walk with no predictable bias (Samuelson 1965), and is defined as available information in relationship to returns. In particular weak, semi strong, and strong forms of market efficiency are analysed. Fama [11] investigates the research question by using serial correlation tests and technical trading rules to test the weak EMH. He mainly observes that the stock market has a weak-form efficiency or that past returns are reflected in the current assets prices. In other words, he argues that current prices are predictable by observing price trends from the past.

There are also extensive investigation in Stock Market Anomalies in terms of value, size and other regularities. Basu [3] documents the use of price/earnings (P/E) ratios to forecast stock returns. He observes that low P/E securities outperform their high P/E counterparts by more than 7% per year. Banz [2] examines the period 1931-75 for shares listed on the New York Stock Exchange. Over this interval, the 50 smallest stocks outperformed the 50 largest stocks by an average of 1% point per month on a risk-adjusted basis. Negative long-run performance of new issues are documented by Ritter [25] and Loughran and Ritter [21]. However, their analyses show that there is a benchmark problem. Later, Fama [12] tests the predictability of prices through estimation of dividend to price ratio (Div/P), earning to price ratio (E/P), book to market value ratio (B/M) and various measures of interest rates. He found spurious results and is skeptical about their relevancy[1].

Beyond the mentioned studies other methods were applied to test the validity of the EMH. Some central validity tests to be mentioned are considered below. Tests of fundamental valuation consider whether prices are confirmed to fundamentals values and whether the price level of securities is correct. Variance bounds are considered as a difficulty to test this hypothesis. The central assumption is that the excess price volatility implies market inefficiency. Shiller [30] examines the variation in stock market prices and finds that price fluctuations are too large to be justified by the subsequent variation in dividend payments.

Tests of overreaction and underreaction of price returns are further important methods. DeBondt et al. [6] observe that return predictability has a positive autocorrelation in security returns over weekly and monthly time horizons; and secondly there is an indication of negative serial correlation in longer horizon returns over periods of several years. They explain this pattern of return reversal as an overreaction in the market in which stock prices diverge from their fundamental value. Another central path of validity tests are based on market microstructure. Treynor [31] generates a model in which informed and uninformed investors are competing, and investors with inside information win. Grossman and Stiglitz [15] analyse transactions from uninformed noise traders. They find that when including information of transaction costs it is impossible for markets to be informationally efficient.

[1]Some divergent results originate from seasonality of stock returns, which are analyzed for example by Arshanapalli (1998).

Efficient market hypothesis became a dominant paradigm during the 1970's, despite the fact that, testing market efficiency is difficult. In conclusion, possible shortcomings are included in the models. On this theoretical background, models concerning portfolio choice, consumption and utility maximization account for lack of information efficiency.

In this Chapter we illustrate model specifications for insider investors in a market where frictions are exemplified by jumps. Oksendahl and Biagini [4] present an insider trading model with a Brownian motion driven market on a filtered probability space. Their optimal portfolio model uses forward integrals to achieve filtration of information. Kohatsu-Higa [17] analyzes the filtration methods and anticipating calculus for utility maximization in optimal insider trading. Elliott et al. [10] examine a similar approach to illustrate the difference in utility maximization between an insider investor and an uninformed investor. Di Nunno et al. [8] introduce Malliavin calculus to utilize forward integration with respect to Poisson random measures.

2. EFFICIENT MARKET HYPOTHESIS

The efficient markets hypothesis (EMH), popularly known as the Random Walk Theory, is the proposition that current stock prices fully reflect available information about the value of the firm, and there is no way to earn excess profits (more than the market overall), by using this information. It deals with one of the most fundamental and exciting issues in finance - why prices change in security markets and how those changes take place. It has very important implications for investors as well as for financial managers. Fama [12] introduced the term "efficient market" in his 1965 paper. He defined it, as a market in which competition will cause, on average, the full effects of new information to be reflected "instantaneously" in actual prices.

Many investors try to identify securities that are undervalued, and are expected to increase in value in the future, and particularly those that will increase more than others. Many investors, including investment managers, believe that they can select securities that will outperform the market. They use a variety of forecasting and valuation techniques to aid them in their investment decisions. Obviously, any edge that an investor possesses can be translated into substantial profits. The EMH asserts that none of these techniques are effective, and therefore no one can predictably outperform the market.

The efficient markets hypothesis (EMH) suggests that profiting from predicting price movements is very difficult and unlikely. The main engine behind price changes is the arrival of new information. A market is said to be "efficient" if prices adjust quickly and, on average, without bias, to new information. As a result, the current prices of securities reflect all available information at any given point in time. Consequently, there is no reason to believe that prices are too high or too low. Security prices adjust before an investor has time to trade on and profit from a new piece of information.

The key reason for the existence of an efficient market is the intense competition among investors to profit from any new information. The ability to identify over- and underpriced stocks is very valuable. Consequently, many people spend a significant amount of time and resources in an effort to detect "mispriced" stocks. Naturally, as more and more analysts compete against each other in their effort to take advantage of over- and undervalued

securities, the likelihood of being able to find and exploit such mispriced securities becomes smaller and smaller. In equilibrium, only a relatively small number of analysts will be able to profit from the detection of mispriced securities, mostly by chance. For the vast majority of investors, the information analysis payoff would likely not outweigh the transaction costs.

At any point in time, prices of securities in efficient markets reflect all known information available to investors. There is no room for fooling investors, and as a result, all investments in efficient markets are fairly priced. Fair pricing of all securities does not mean that they will all perform similarly, or that even the likelihood of rising or falling in price is the same for all securities. According to capital markets theory, the expected return from a security is primarily a function of its risk. The price of the security reflects the present value of its expected future cash flows, which incorporates many factors such as volatility, liquidity, and risk of bankruptcy.

However, while prices are rationally based, changes in prices are expected to be random and unpredictable, because new information, by its very nature, is unpredictable. Therefore stock prices are said to follow a random walk.

3. JUMP PROCESSES

In the real world, the asset price processes have jumps and the empirical distribution of asset returns exhibits fat tails and skewness that deviates from normality. Therefore, in the risk neutral world, the major deficiency of the Black and Scholes model is the normality assumption of logarithmic across all time scales and assumption of non-stochastic volatility. Since the classical Black and Scholes model makes these strong assumptions, it does not give appropriate modelling of stochastic behaviour of stock prices and indices over time. On the other hand, a Lévy processes are a simple Markov process with jumps. They provide us an appropriate framework for modelling the movements of financial assets as stochastic processes and capture a huge class of asset returns. Lévy processes have shown to provide a good fit with real data while the diffusion models can not consider the discontinuous sudden movements observed on asset prices (see Onalan [24]).

3.1. Lévy Processes: Definitions And Properties

The term Lévy process honours the work of the French mathematician Paul Lévy (1886-1971). A Lévy process is constructed by giving a distribution on a Euclidean space with the property of infinite divisibility and the technical tools are provided by the theory of stochastic processes with independent and stationary increments distributed according to the given distribution [24].

Definition 1

A stochastic process $X = \{X_t\}_{t\in[0,\infty)}$ defined on the probability space (Ω, F, P) is said to be a Lévy Process if it possesses the following properties:

1) The paths of X are P-almost surely right continuous with left limits,

2) $P(X_0 = 0) = 1$, i.e, $X_0 = 0$ a.s.,
3) $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s},
4) $0 \leq s \leq t$, $X_t - X_s$ is independent of $X_u : u \leq s$.

Definition 2

A probability distribution F on $\mathbb{R}^d$ is said to be infinitely divisible if for any integer $n \geq 2$, there exists n independent and identically distributed (i.i.d.) random variables $Y_1, Y_2, \ldots, Y_n$ such that each of them has distribution of F.

Proposition 1

If X_t is a Lévy process, then X_t is infinitely divisible for each $t \geq 0$.

One way to establish whether a given random variable has an infinitely divisible distribution is via its characteristic exponent. The full extent to which we may characterise infinitely divisible distributions is described by the characteristic exponent and an expression known as the Lévy - Khintchine formula.

Definition 3: (Lévy - Khintchine Formula)

The probability law or distribution P of a real valued random variable X is infinitely divisible iff there exists a triplet (b, c, ν) with $b \in \mathbb{R}$ (drift term), $c \in \mathbb{R}^+$ (diffusion coefficient) and a measure ν (Lévy measure) satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(dx) < \infty$ (integrability condition), such that

$$\phi_{X_t}(u) = E(e^{iuX_t}) = e^{i\Psi(u)}, \tag{1}$$

and

$$\Psi(u) = ibu - \frac{1}{2}u^2c + \int_{\mathbb{R}} (e^{iux} - 1 - iuxI_{|x|<1})\nu(dx). \tag{2}$$

As a result of Lévy-Khintchine Formula, for any Lévy process $\{X_t\}_{t\geq 0}$ there exists a continuous function $\Psi(u): \mathbb{R}^d \to \mathbb{R}$ such that

$$E(e^{iuX_t}) = e^{i\Psi(u)} = e^{i(ibu - \frac{1}{2}u^2c + \int_{\mathbb{R}} (e^{iux} - 1 - iuxI_{|x|<1})\nu(dx))}, \tag{3}$$

where $u \in \mathbb{R}^d$ and $\Psi(u)$ is the characteristic exponent of random variable X_t.

To conclude our introduction to Lévy processes and infinite divisible distributions, let us proceed to some concrete examples.

3.2. Some Examples of Lévy Processes

3.2.1. The Poisson Process

For each $\lambda > 0$ consider a probability distribution P_λ is said to be Poisson distribution, if it is concentrated on $k = 0,1,2, \ldots$, such that $P_\lambda(k) = e^{-k}\frac{\lambda^k}{k!}$. The characteristic function of the sum of the n independent Poisson processes, each of which with parameter λ/n is

$$\sum_{k\geq 0}\ (e^{iuk}P_\lambda(k)) = e^{-\lambda(1-e^{iu})} = (e^{\frac{\lambda}{n}(1-e^{iu})})^n. \tag{4}$$

Recall that a Poisson process, $\{N_t\}_{t\geq 0}$ is a Lévy process with distribution at time $t \geq 0$ which is Poisson with parameter λt. From the above calculations we have $E(e^{iuN_t}) = e^{-\lambda t(1-e^{iu})}$ and hence its characteristic exponent is given by $\Psi(u) = \lambda(1 - e^{iu})$.

3.2.2. The Compound Poisson Processes

For financial applications, it is of little interest to have a process with a single possible jump size. The compound Poisson process is a generalization where the waiting times between jumps are exponential but the jump sizes can have an arbitrary distribution. More precisely, let N be a Poisson process with parameter λ and $\{Y_j\}_{j\geq 1}$ be a sequence of independent random variable with law f. The process $X_t = \sum_{j=1}^{N_t} Y_j$ is called compound Poisson process. Its trajectories have RCLL (right continuous and left limit) and are piecewise constant but the jump sizes are now random with law f. The compound Poisson process has independent and stationary increments. Its law at a given time t is not known explicitly but the characteristic function is known and has the form

$$E(e^{iuX_t}) = e^{-\lambda t\int_{\mathbb{R}}(1-e^{iux})f(dx)}. \tag{5}$$

3.2.3. Jump-Diffusions and Lévy Processes

Combining a Brownian motion with drift and a compound Poisson process, we obtain the simplest case of a jump-diffusion, a process which sometimes jumps and has a continuous but random evolution between the jumps times:

$$X_t = \mu t + \sigma W_t + \sum_{j=1}^{N_t} Y_j. \tag{6}$$

The best known model of this type in finance is the Merton model, where the stock price is $S_t = S_0 e^{X_t}$ with X_t having above distribution and jumps Y_j have Gaussian distribution. The process above is again a Lévy process and its characteristic function can be computed by multiplying the characteristic function of the Brownian motion and that of the compound Poisson process (since the two parts are independent):

$$E(e^{iuX_t}) = e^{t(i\mu u+\frac{\sigma^2u^2}{2}+\lambda\int_{\mathbb{R}}(e^{iux}-)f(dx)}. \tag{7}$$

There are many more known examples of infinitely divisible distributions and hence associated Lévy processes. For a comprehensive overview see [7, 20].

Now, we would like to give a brief introduction to Stochastic Hybrid Systems (SHS) and then move on to explore the implications of the mentioned theory in order to turn our attention to the applications of Lévy processes and SHS.

4. STOCHASTIC HYBRID SYSTEMS (SHS)

4.1. Hybrid Systems

Hybrid Systems are dynamical systems that are the combination of both continuous and discrete behaviour. In other words, in a typical hybrid system, the state space is hybrid in the sense that some components take values in a Euclidean space while some other components are discrete. As a result, the methodology of these systems are based on the principles and results of both continuous and discrete systems theories which makes their analysis quite involved and challenging.

In order to evaluate the Hybrid Systems, mathematical models are needed to combine the dynamics of the continuous parts of the system with the dynamics of discrete parts. These mathematical models consist of some forms of the differential or difference equations and some discrete-event models. Consequently, one way to look at an hybrid system is to consider it as an extension of a classical time-driven system by discrete events causing a change in its dynamic behaviour.

A time-driven system may be defined by a differential equation

$$\dot{x} = f(x, u, t), \tag{8}$$

where $x \in \mathbb{R}^n$ denotes the state variable, $u \in \mathbb{R}^m$ is a parameter (which can be used as control), $t \in \mathbb{R}$ denotes time and $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth enough map.

A time-driven system may also be modelled by a difference equation

$$x_{n+1} = f(x_n, u_n, t), \tag{9}$$

where the quantities $x_n \in \mathbb{R}^n, u_n \in \mathbb{R}^m$ and $n \in \mathbb{Z}$.

Then, the time driven systems above can be used to obtain a hybrid system by letting the map depend also on a stochastic process α_t on a discrete state space M which represents the switching between the distinct qualitative behaviours or distinct time-drive systems. For example, a Markov chain on M for modelling a stochastic process such as switching. One possible source of randomness can be represented as a stochastic process α_t that models uncontrollable events that occur at random points in time. Another way to model the randomness in the system may be represented as "noise" in one or more time-driven components of the system by replacing the differential and difference equation of the time-driven systems above by their stochastic analogues, leading to the more general class of Stochastic Hybrid Systems. Basically, this is the setup for the study of asset prices dynamics, which may be modelled by means of Stochastic Differential Equations. The random fluctuations in the evaluation of the asset prices may be a consequence of uncertainty

regarding the initial condition, as well as of the noise generated by either a Brownian motion or a Lévy process.

A typical example of a stochastic differential equations is given by

$$dS_t = \mu(t, S_t, u)dt + \sigma(t, S_t, u)dX_t, \tag{10}$$

with $S_0 = Z$, where $\mu: [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector of the assets appreciation rates and $\sigma: [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{n+k}$ is the matrix of the assets volatilities, the initial condition Z is an $\mathbb{R}^n$-valued random vector and the continuous time stochastic process $\{X_t\}_{t\geq 0}$ denotes either a k-dimensional Brownian motion or Lévy process. The existence of such $\mathbb{R}^n$-valued stochastic process $\{S_t\}_{t\geq 0}$ which is a solution of the stochastic differential equation above can be guaranteed under suitable conditions on the coefficient functions [18].

4.2. Models of Hybrid Systems

Due to the hybrid structure of the system, the following elements have to be defined for every model of Hybrid Systems:

1) $\mathbb{Q}$ is the discrete state space, such as $\mathbb{Q} = \{1,2,\dots,N\}$,
2) X is the continuous state space, such that $X = \mathbb{R}^n$,
3) f is a set of vector field describing the continuous dynamics for all $q \in \mathbb{Q}$,
4) I is a set of initial values (q_0, x_0) of the hybrid state,
5) δ is the discrete state transition function that describes the change of the state space,
6) G is a set of guards prescribing when a discrete state transition occurs. A guard describes a region in the state space X. If the state x is in this region, a discrete state transition may occur.

Definition 4: (Stochastic Hybrid System (SHS))

A stochastic hybrid system is a collection $H = (\mathbb{Q}, X, Inv, f, g, G, R)$
where:

1) $\mathbb{Q}$ is a discrete state space, including discrete variables taking countably many values in $\mathbb{Q} = \{1,2,\dots,N\}$,
2) X is the continuous state space, including continuous variables taking values in $X = \mathbb{R}^n$ for some n,
3) $Inv: \mathbb{Q} \to 2^X$ assigns to each $q \in \mathbb{Q}$ an invariant open subset of X,
4) $f, g: \mathbb{Q} \times X \to TX$ are vector fields,
5) $G: E = \mathbb{Q} \times \mathbb{Q} \to 2^X$ assigns to each $e \in E$ a guard $G(e)$ such that:
   a) For each $q \in \mathbb{Q}$, the family $G(e): e = (q, q')$ for some $q' \in \mathbb{Q}$ is a disjoint partition of $\partial Inv(q)$,
   b) For each $e = (q, q') \in E$, $G(e)$ is a measurable subset of $\partial Inv(q)$ (possible empty).

6) $R: E \times X \to P(X)$ assigns to each $e = (q, q') \in E$ and $x \in G(e)$ a reset probability kernel on X concentrated on $Inv(q')$.

Here $P(X)$ denotes the family of all probability measures on X. Furthermore, for any measurable set $A \subset Inv(q')$, $R(e,r)(A)$ is a measurable function in X.

The measurability assumption on R in the preceding definition is made to ensure that the events we encounter later are measurable with respect to the underlying σ-field, hence their probabilities make sense.

Definition 5: (Stochastic Execution)

A stochastic process $(X_t, Q_t) \in X \times \mathbb{Q}$ is called stochastic execution iff there exists a sequence of stopping times $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \ldots$ such that for each $n \in N$ the following holds:

1) In each interval $[\tau_n, \tau_{n+1})$, $Q_t = Q_\tau$ is constant, X_t is a continuous solution to the Stochastic Dynamic Equation (SDE):

$$dX_t = f(Q_{\tau_n}, X_t)dt + g(Q_{\tau_n}, X_t)dW_t, \tag{11}$$

where W_t is the standard Brownian Motion in R,

2) $\tau_{n+1} = \inf\{t \geq \tau_n : X_t \notin Inv(Q_{\tau_n})\}$,
3) $X_{\tau_n^-} \in G(Q_{\tau_n}, Q_{\tau_n+1})$ where $X_{\tau_n^-}$ denotes $\lim_{t \uparrow \tau_{n+1}} (X_t)$,
4) The probability distribution of X_{τ_n+1} given $X_{\tau_{n+1}^-}$ is governed by the law $R(e_n, X_{\tau_{n+1}^-})$, where $e_n = (Q_{\tau_n}, Q_{\tau_n+1}) \in E$.

4.2.1. Deterministic and Non-deterministic Hybrid Systems

There is much work that has focused on the deterministic models of hybrid systems, however, in practice, it is often quite desirable to introduce uncertainty in the models. To address this need, researchers introduced non-deterministic models by allowing, for example, undermodelling of certain parts of the system, external unmodelled disturbances, etc. While the deterministic models characterize the future of the system without uncertainty, the non-deterministic models allow uncertainty to enter in some different places of the system as continuous evolution choice, discrete transition destination choice, choice between continuous evolution and a discrete transition, and so on. Both models can capture a wide range of applications, however, they have limitations. For instance, non-deterministic systems provide no way of distinguishing between their many possible solutions. This implies that only worst-case analysis is possible with non-deterministic hybrid systems, since they only let us only pose yes-no type (qualitative) questions. Another specific problem in the area of stochastic hybrid control is the stochastic reachability problem, that is, computing the probability that the system state will remain in a given subset of the state space [5, 22].

4.2.2. Continuous-Time and Discrete-Time Stochastic Hybrid Systems

In the model proposed by Hu et al. at. [16], the deterministic differential equations for the continuous flows are replaced by their stochastic counterparts, and the reset maps are generalized to state-dependent distributions that define the probability density of the state after a discrete transition. However, in this model transitions are always triggered by deterministic conditions on the state. Transitions between modes are triggered by stochastic events, much like the transitions between states of a continuous-time Markov chains. However, the probability that a transition occurs in a given interval of time depends both on the continuous and discrete components of the current SHS's state.

5. PORTFOLIO OPTIMIZATION MODEL FRAMEWORK

Okur and Davis [23] developed and identified some necessary conditions and found the optimal strategy by using forward integral techniques for the portfolio optimization model with filtration of insider information. They examined a stochastic optimal control problem in a financial market driven by a Lévy process with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T)}$, by assuming that there are two kinds of investors with different levels of information in the market: an uninformed agent whose information coincides with the natural filtration of the price processes and an insider who has more information than the uninformed agent.

5.1. Stochastic Hybrid System for Portfolio Optimization

The model uses in particular a specification of the Stochastic Hybrid Model type introduced by Gosh and Bagchi [1]. Forward integration with respect to the Brownian motion is related to Russo and Vallois [26, 27, 28], in the case of forward integration with respect to Poisson random measure, the methodology is associated with DiNunno [8],

$$\max_{(c,\pi)} \mathbb{E}\left[\int_0^T e^{-\delta_t} \ln c_t dt + K e^{-\delta_T} \ln X_T^{(c,\pi)}\right]. \tag{12}$$

$(X_t, \theta_t)_{t\in[0,\infty)}$ is a switching jump-diffusion between jumps, where $X_t \in \mathbb{R}^d$, $\theta_t \in \Theta = \{0,1,\dots,N\}$, with θ_t being a pure jump process taking values in Θ.

5.1.1. Preliminary Description

The portfolio optimization problem and asset returns follow the dynamics (for $t \geq 0$) which can be formulated as:

$$dX_t = \mu(X_t, \theta_t)dW_t + \int_{\mathbb{R}} g(X_{t^-}, \theta_{t^-}, u)N(dt, du), \tag{13}$$

$$d\theta_t = \int_{\mathbb{R}} h(X_{t^-}, \theta_{t^-}, u)N(dt, du). \tag{14}$$

The relative consumption rate is defined as follows:

$$\lambda_t := \frac{c_t}{X_t^{(c,\pi)}}. \tag{15}$$

The relative investment rate with risk aversion is given in the following expression:

$$\pi_t := \frac{X^{\lambda}-1}{\lambda}. \tag{16}$$

It is assumed that logarithmic utility function has the following specification:

$$U(x,t) := \frac{X^{1-\gamma}-1}{1-\gamma}, \tag{17}$$

$X(0) = X_0$, $\theta(0) = \theta_0$. The price process X_t switches from one jump-diffusion path to another as the discrete component θ_t moves from one step to another.

Function h: $\mathbb{R}^d \times \Theta \times \mathbb{R} \to \mathbb{R}^N$ is defined as:

$$h(x, e_i, u) = \begin{cases} j-i & \text{if } u \in \Delta_{ij}(x), \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Bagchi and Ghosh [1] proved that under the following conditions there exists a unique strong solution of Stochastic Dynamic Equations (SDE):

$$dX_t = \alpha(X_t, \theta_t)dt + b(X_t, \theta_t)dW_t + \int_{\mathbb{R}} g(X_{t^-}, \theta_{t^-}, u)p(dt, du), \tag{19}$$

$$d\theta_t = \int_{\mathbb{R}} h(X_{t^-}, \theta_{t^-}, u)p(dt, du), \tag{20}$$

1) For each $i = 1,2,\ldots,N$, $\alpha(\cdot, i)$ is bounded and Lipschitz continuous,
2) For each $i = 1,2,\ldots,N$, $b(\cdot, i)$ is bounded and Lipschitz continuous,
3) For all $i, j \in \{1,\ldots,N\}$, $i \neq j$ functions $\lambda_{ij}(\cdot)$ are bounded and measurable, $\lambda_{ij}(\cdot) \geq 0$ for $i \neq j$ and $\sum_{j=1}^{N} \lambda(\cdot) = 0$ for any $i \in \{1,\ldots,N\}$,
4) U_i, the projection of support of $g(\cdot,\cdot,\cdot)$ on $\mathbb{R}$ is bounded.

$N(\cdot)$ is the Poisson random measure, $\widetilde{N}(\cdot)$ is the Poisson process, and (Ω, F, P) is a product probability space corresponding to the given Poisson random measure, where

$$(\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \otimes P_\eta). \tag{21}$$

The standard Brownian motion is defined as $\{W_t\}_{0 \leq t \leq T}$, and pure jump Lévy process of $\{\eta_t\}_{0 \leq t \leq T}$ is defined as,

$$\eta_t = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(dt, dz), \tag{22}$$

with, $\widetilde{N}(dt,dz) = (N - \nu F)(dt,dz) = N(dt,dz) - \nu F(dz)dt$, and $D_N(\cdot)$ being a domain of $\widetilde{N}(\cdot)$, $D = \{t \in D_{\widetilde{N}} | \widetilde{N}(t), \in U\}$, B denotes the Borel sets, and $\mathcal{F} = \sigma\{W_s, N(A,B) | s \leq t, A \in B([0,T]), B \in B(\mathbb{R})\}$ is the augmented filtrations generated by $W(\cdot)$ and $N(\cdot,\cdot)$. This filtration is given as $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}$, $\forall t \in [0,T]$.

Definition 6

Let $\varphi(\cdot,\omega), \omega \in \Omega_B$ be a measurable process. The forward integral of φ with respect to Brownian motion is defined by

$$\int_0^\infty \varphi(t,\omega) d^- W_t = \lim_{\varepsilon \to 0} \int_0^\infty \varphi(t,\omega) \frac{W_{t+\varepsilon} - W_t}{\varepsilon} dt, \tag{23}$$

if the limit exists (uniformly in t) in probability. The variable φ is then called forward integrable with respect to Brownian motion.

Under boundedness, measurably and Lipschitz conditions a pathwise unique solution of (13-14) exists.

Definition 7

Let $\varphi(\cdot,z,\omega)$ be measurable random field. The forward integral of φ called forward integrable with respect to compensated Poisson random measure is defined by

$$\int_0^\infty \int_{\mathbb{R}_0} \varphi(t,z) \widetilde{N}(d^- t, dz) = \lim_{n \to \infty} \int_0^\infty \int_{U_n} \varphi(t,\omega) \widetilde{N}(d^- t, dz), \tag{24}$$

where U_n is an increasing sequence of compact sets with $U_n \subseteq \mathbb{R}_0$, $\nu_F(U_n) < \infty$ and $\cup_{n=1}^\infty U_n \subseteq \mathbb{R}_0$.

Considering the SDE $Y_t = X_0 + \int_0^t \mu(Y_s, \theta_0) ds + \int_0^t \sigma(Y_s, \theta_0) dW_s$, the unique solution $\{X_1(t), \theta_1(t)\}_{t \in [0,\tau_1]}$ in time interval $[0,\tau_1]$ is found as

$$X_1(t) = \begin{cases} Y_t & \text{if } \ 0 \leq t < \tau_1, \\ Y_{\tau_1^-} + g(Y_{\tau_1^-}, \theta_0, \widetilde{N}(\cdot)) & \text{if } \ t = \tau_1, \end{cases} \tag{25}$$

$$\theta_1(t) = \begin{cases} \theta_0 & \text{if } \ 0 \leq t < \tau_1, \\ Y_{\tau_1^-} + g(Y_{\tau_1^-}, \theta_0, \widetilde{N}(\cdot)) & \text{if } \ t = \tau_1. \end{cases} \tag{26}$$

6. Optimal Investment and Consumption Portfolio

This Section provides an application of the model in Section 5. We introduce stochastic hybrid dynamics which is described in Bagchi and Ghosh [1] into the model in (12) with two types of investors who optimize their consumption and investment plans.

6.1. Problem Statement

An investor with a finite lifetime must determine the amounts c_t that will be consumed and the fraction of this wealth π_t invested in stock portfolio, so as to maximize expected lifetime utility.

The wealth process evolves with the following SDE:

$$d^{-}X^{(c,\pi)}(X_t,\theta_t) = X^{(c,\pi)}(X_t,\theta_t)[\{r_t - \lambda(X_t,\theta_t) + \mu(X_t,\theta_t - r_t)\pi(X_t,\theta_t)\}dt + \\ +\sigma(X_t,\theta_t)\pi(X_t,\theta_t)d^{-}B_t + \pi_t \int_{\mathbb{R}_0} g(X_{t^-},\theta_{t^-},u)N(dt^{-},du)]. \quad (27)$$

The objective is the investor's utility function with the assets dynamics as the given constraints:

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & E\left[\int_0^T U(c_s,s)ds + g(X_T^{(c,\pi)},T)\right] \\ \text{subject to} \quad & J(t) \geq 0, \\ & X_t > 0 \text{ for } t \in [0,T], \\ & X_0 = \upsilon. \end{aligned} \quad (28)$$

Assuming a logarithmic utility function, which is iso-elastic, the optimization problem becomes:

$$J(v,t) = \sup_{(J,\pi)} E\left[\int_0^T e^{-\delta_s}U(P_s,s)ds + e^{-\delta_T}U\left(X_T^{(P,\pi)},T\right)\right]. \quad (29)$$

We assume a constant relative risk aversion (CRRA) utility function $U(x,t) = \frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma \in (0,\infty)\backslash\{1\}$. Thus, the objective function takes the following form:

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & E\left[\int_0^T e^{-\delta_s}U(P_s,s)ds + e^{-\delta_T}U(X_T^{(P,\pi)},T)\right] \\ \text{subject to} \quad & c(t) \geq 0, \\ & X_t > 0 \text{ for } t \in [0,T], \\ & X_0 = \upsilon. \end{aligned} \quad (30)$$

6.1.1. Optimal Investment and Consumption Portfolio

To derive a solution by the application of dynamic programming the corresponding Hamilton-Jacobi-Bellman (HJB) equation is given as:

$$\rho J(v,t) = \max_{c_t \geq 0, \pi_t \in \mathbb{R}^*}\{U(c_t) + J_v(v,t) + J(v,t)(v[\pi_t^T(\mu - r) + r] - c_t) \\ +\frac{1}{2}J_{vv}(v,t)v^2\pi_t^T\sigma\sigma^T\pi_t\}. \quad (31)$$

The related static optimization problem can be formulated as follows:

$$\max_{c_t \geq 0, \pi_t \in \mathbb{R}^*}\{U(c_t) + J_v(v,t) + J(v,t)(v[\pi_t^T(\mu - r) + r] - c_t)$$

$$+\frac{1}{2}J_{vv}(v,t)v^2\pi_t^T\sigma\sigma^T\pi_t\}. \tag{32}$$

The corresponding Lagrange function is given as:

$$L\big(\pi(v,t),c(v,t),\tilde{\lambda}(v,t)\big)=J_v(v,t)(v[\pi_t^T(\mu-r)+r]-c_t) \\ +\frac{1}{2}v^2|\pi_t^T\sigma|^2J_{vv}(v,t)+U(c_t)-\tilde{\lambda}(v,t)c_t. \tag{33}$$

As Okur and David [23] determined, the relative consumption rate can be chosen independently from the optimal portfolio if the terminal wealth is added to the problem when the agent has logarithmic utility function. Samuelson [29] defines the initial optimal consumption rate prior to information, for all $t \geq 0$, as:

$$\hat{\lambda}_t := \frac{e^{-\delta_t}}{\int_t^T e^{-\delta_s}ds+Ke^{-\delta_T}}. \tag{34}$$

Substituting back into the optimality condition, the Lagrangian function takes the following form:

$$-\delta J(v,t)+\frac{(c(v,t))^{1-\gamma}}{1-\gamma}+J_v(v,t)+J(v,t)(v[\pi(v,t)_t^T(\mu-r)+r] \\ -c^{opt}(v,t))+\frac{1}{2}J_{vv}(v,t)v^2(\pi(v,t))^T\sigma\sigma^T(\pi^{opt}(v,t)). \tag{35}$$

6.2. Optimal Investment and Consumption Portfolio

6.2.1 Brownian Case: Optimal Investment and Consumption Portfolio

Under the conditions that $\gamma(t,z)=0$ and $\sigma(t)\neq 0$ for almost all (t,z) the following optimal results for the insider are obtained:

$$\pi_i^*(t)=\frac{\mu_t-r_t}{(\sigma_t)^2}+\frac{\alpha_t}{\sigma_t}, \quad \forall t\in[0,T], \tag{36}$$

and the relative consumption is given as:

$$\lambda_i^*(t)=\frac{e^{-\delta_t}}{\int_t^T e^{-\delta_s}ds+Ke^{-\delta_T}}. \tag{37}$$

In the case of an uninformed agent the results are given in the following expressions:

$$\pi_h^*(t)=\frac{\mu_t-r_t}{(\sigma_t)^2}, \lambda_h^*(t)=\frac{e^{-\delta_t}}{\int_t^T e^{-\delta_s}ds+Ke^{-\delta_T}}=\lambda_i^*(t). \tag{38}$$

The optimal wealth process is obtained as:

$$J_i(c^*_{\lambda_i}, \pi^*_i) = J_h(c^*_{\lambda_h}, \pi^*_h) + \frac{1}{2}\mathbb{E}\left[\int_0^T e^{-\delta_t}\int_0^T (\alpha_s)^2 ds\, dt\right] + \frac{1}{2}K\,\mathbb{E}\left[e^{-\delta_T}\int_0^T (\alpha_s)^2 ds\right]. \tag{37}$$

6.2.2. Mixed Case: Optimal Investment and Consumption Portfolio

Under the conditions that $\gamma(t,z) = z$ and $\sigma(t) \neq 0$ for almost all (t,z), and also applying the filtration as described in Section 5.1, which are:

1) The informed agent has access to the filtration $\mathcal{G}_t$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t, t \in [0,T]$,
2) The Lévy measure v_F is given by $v_F(ds,dz) = \rho\delta_1(dz)ds$ where $\delta_1(dz)$ is the unit point mass at 1,
3) η_t is defined as $\eta_t = Q(t) - \rho t$ with Q being a Poisson process of intensity ρ.

In the mixed case and with an insider the optimal solution is determined as follows:

$$\pi^*_i(t) = \pi^*_h(t) + \frac{\zeta_t}{\sigma_t}, \tag{38}$$

with

$$\pi^*_h(t) = \frac{\mu_t - r_t}{(\sigma_t)^2} - \frac{\rho}{(\sigma_t)^2}, \tag{39}$$

$$\zeta_t = \frac{1}{2\sigma_t}\left[-\mu_t + r_t + \rho + \sigma_t\alpha_t - (\sigma_t)^2 + \sqrt{(\mu_t - r_t - \rho + \sigma_t\alpha_t + {\sigma_t}^2)^2 + 4(\sigma_t)^2\theta_t}\right], \tag{40}$$

$$\alpha_t = \frac{\mathbb{E}\left[B_{T_0} - B_s | G_s\right]^-}{T_0 - s}, \tag{41}$$

$$\theta_t = \frac{\mathbb{E}\left[Q_{T_0} - Q_s | G_s\right]^-}{T_0 - s}, \tag{42}$$

where expression $\mathbb{E}[\cdot]^-$ denotes the left limit in s.

Optimal consumption rates are equally expressed for both types of investors:

$$\lambda^*_i(t) = \frac{e^{-\delta_t}}{\int_t^T e^{-\delta_s}ds + Ke^{-\delta_T}} = \lambda^*_h(t). \tag{45}$$

The resulting wealth process of the informed investor is described as follows:

$$\begin{aligned} J_i(c^*_{\lambda_i}, \pi^*_i) = J_h(c^*_{\lambda_h}, \pi^*_h) &+ E\left[\int_0^T e^{-\delta_t}\int_0^t (-\frac{1}{2}{\zeta_s}^2 + \zeta_s\alpha_s)ds\, dt\right] \\ &+ E\left[\int_0^T e^{-\delta_t}\int_0^t \int_{\mathbb{R}_0}^T \ln\left(1 + \frac{z\zeta_s\sigma_s}{{\sigma_s}^2 + (\mu_s - r_s - \rho)z}\right) v_G(ds,dz)\, dt\right] \\ &+ K\,E\left[e^{-\delta_T}\int_0^T (-\frac{1}{2}{\zeta_s}^2 + \zeta_s\alpha_s)\, ds\right] \\ &+ K\,E\left[e^{-\delta_T}\int_0^T \int_{\mathbb{R}_0}^T \ln\left(1 + \frac{z\zeta_s\sigma_s}{{\sigma_s}^2 + (\mu_s - r_s - \rho)z}\right) v_G(ds,dz)\right]. \end{aligned} \tag{43}$$

7. PORTFOLIO OPTIMIZATION MODEL

This Section provides a slight generalization of the model in Section 6. We consider a financial market where the risk-free asset price $\{S_t^0\}_{0\le t\le T}$ is satisfying following dynamic:

$$dS_t^0 = r(t, X_t, \theta_t)dt, \tag{44}$$

with the initial condition of $S_0^0 = s_0$. The case of $N \ge 1$ risky assets with $\{S_t^n\}_{0\le t\le T, 1\le n\le N}$ prices is given in the following dynamics:

$$dS_t^n = \mu_n(t, X_t)S_t^n dt + S_t^n \sum_{m=1}^{M} \sigma_{n,m}(t, X_t)dW_t^m, S_0^n = s_n > 0. \tag{45}$$

As described in Section 5.1.1, we assume that $N(\cdot)$ is the Poisson random measure, $\widetilde{N}(\cdot)$ is the Poisson process, and (Ω, F, P) is a product probability space corresponding to the given Poisson random measure, where

$$(\Omega, P) = (\Omega_B \times \Omega_\eta, P_B \otimes P_\eta). \tag{46}$$

The Brownian motion is defined as $W_t = (W_t^1, \dots, W_t^M)^T$ which is a standard M-dimensional Brownian motion. Moreover, $r(t, x)$ is the risk free interest rate, $\sigma(t, x)_{n,m} = (\sigma_{n,m}(t, x))_{1\le n\le N, 1\le m\le M}$ is the matrix of risky assets' volatilities, and X_t is the state of a Markov process $\{X_t\}_{0\le t\le T}$ on a finite state space $S = \{s_1, .., s_n\}$ at time t.

The Markov process transitions occur only on integer values of time t, which makes the Markov process an extension to continuous time of a Markov chain on S. In this regard, $r(t, x)$, $\mu(t, x)$, and $\sigma(t, x)$ are deterministic continuous functions on $[0, T]$ for every $s \in S$. The asset volatility matrix $\sigma(t, x)\sigma^T(t, x)$ is non-singular with respect to Lebesgue measure for almost all $t \in [0, T]$ and satisfies the integrability condition of

$$\sum_{n=1}^{N} \sum_{m=1}^{M} \int_0^T \sigma_{n,m}^2(t, x)dt < \infty. \tag{47}$$

The consumption process $\{c_t\}_{0\le t\le T}$ is a progressively measurable non-negative process such that $\int_0^t c_t dt < \infty$. The fraction of the investor's wealth allocated to the asset S^n at time t is given as $(\pi_n(t))_{0\le t\le T}$. So, the portfolio process takes the following form:

$$\Pi(t) = (\pi_0(t), \pi_1(t), \dots, \pi_N(t))^T, \tag{48}$$

where $\sum_{n=0}^{N} \pi_n(t) = 1$, $(t \in [0, T])$. Let $\{\eta\}$ be a process defined by

$$\eta_t = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(dt, dz), \tag{49}$$

where $\widetilde{N}(dt,dz) = (N - \nu F)(dt,dz) = N(dt,dz) - \nu F(dz)dt$. The variable $\{\eta\}$ is a piecewise constant process that changes from n to m when (X_t, θ_t), jumps from the regime $S^n \times M_n$ to the regime $S^m \times M_m$. Thus, it is an indicator of a regime and a change in means of a switching in the regimes in which $\{X_t, \theta_t\}$ evolves. $D_N(\cdot)$ is the domain of $\widetilde{N}(\cdot)$ described as $D = \{t \in D_{\widetilde{N}} | \widetilde{N}_t, \in U\}$, $\mathfrak{B}$ denotes the σ-fields of Borel sets, where $\mathcal{F} = \sigma\{W_s, N(A,B) | s \leq t, A \in \mathfrak{B}([0,T]), B \in \mathfrak{B}(R)\}$. The augmented filtration is generated by $W(\cdot)$ and $N(\cdot,\cdot)$, whereas the filtration is given as $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \forall t \in [0,T]$. The coefficients satisfy the following conditions:

1) $i(\cdot), c(\cdot,\cdot), \pi(\cdot)$ and $\sigma(\cdot,\cdot)$ are $\mathcal{F}$-adapted caglad processes,
2) $\theta(t,z) > -1$ with respect to $dt \times \nu_F(dz)$.

The agent wants to maximize its expected intertemporal utility of consumption and terminal wealth of the portfolio. Hence, we assume that the informed agent's portfolio and consumption choices are adapted to larger filtration $\mathbb{G}$. The filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,T]}$ can not be freely chosen. For example, if $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X)$, is a hedgeable F_T- measurable random variable, then the informed agent will immediately obtain the arbitrary opportunity[2]. The stochastic control problem and characterization of the controls under the filtration $\mathbb{G}$ are given next. Considering the uniqueness conditions of the solution we refer to Section 5.1.1. The wealth process, A_t, is defined as follows:

$$A_t = A_0 + \sum_{n=0}^{N} \int_0^t \frac{\pi_n(s)A_s}{S_s^n} dS_s^n + \int_0^t (i_s - c_s)ds, \tag{50}$$

where A_0 denotes investor's initial wealth, and $i(t)$ is the income at time t. The incorporated deterministic function is described as:

$$\int_0^T i_t dt < \infty.$$

The corresponding wealth process $A(t)$ of the insider with respect to the consumption-income portfolio is governed by the following Stochastic Hybrid equation:

$$\begin{aligned} dA_t = \big(i_s - c_s + \big(\pi_0(t)r(t,X_t) + \textstyle\sum_{n=1}^{N} \pi_n(t)\mu_n(t,X_t)\big)A_t\big)dt + \\ + \textstyle\sum_{n=1}^{N} \pi_n(t)A_t \sum_{m=1}^{M} \sigma_{n,m}(t,X_t)dW_t^m, \end{aligned} \tag{51}$$

for $t \in [0,T]$ and initial value $X_0 = x$.

We assume that the agent has a logarithmic utility function, thus, the assumption is made that the agent has the same relative risk-tolerance until the end of its life.

[2]For further information, see Okur [23].

7.1. Optimization Problem

The investor faces the problem of finding the strategies that maximize i) the utility of consumption for all $t \in [0,T]$ and ii) the terminal wealth at time T.

$$V(x) = \sup_{v \in a_x} E^S_{0,S_0} E^P_{0,w} \left[\int_0^T U_1(c_s, s)ds + U_2(W_T) \right], \tag{52}$$

where a_x is the set of all admissible decision strategies.

$$V(i^*, c^*) = \sup_{(i,c)} V(i^*, c^*). \tag{53}$$

Define c_s as in (54). Then $\hat{c}_s$ is an optimal relative consumption rate independent of the portfolio chosen, in the sense that

$$V(\hat{c}, i) \geq V(c, i), \tag{54}$$

for all c, i such that $(c,i),(\hat{c},i) \in A$. Suppose there exists an optimal pair (c,i) for the given problem, then for all $t \geq 0$,

$$i^*{}_t = \hat{\imath}_t, \tag{55}$$

where $\hat{\imath}_t$ indicate an optimal investment decision independent of the portfolio choice.

We refer to Section 6 for the solution to this portfolio optimization problem. To achieve the final solution, the optimal consumption and investment rates are inserted in the Stochastic Hybrid wealth equation in (56).

8. Risk Insurance Dynamic Programming

The explained models in Sections 5 and 6 can be applied to optimal investment and reinsurance problems with insider information. The insurer with insider information faces a claim process that is modeled by a Brownian motion with drift and has the possibility to reduce the risk involved with this process by purchasing proportional reinsurance coverage. The optimal investment and proportional reinsurance decision for both firms is determined by the solution of suitable expected utility maximization problems, explicitly taking into account their different information sets. Suppose that we have a financial market on the fixed time horizon $[0,T]$ with $T > 0$, consisting of a riskless asset (bond) and a risky one (stock), with dynamics given by

$$dS_t^0 = rS_t^0 dt, \tag{56}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{57}$$

where S_t^0 is the price of the bond and S_t the price of the stock at time t, with initial conditions $S_0^0 = 1$ and $S_0 > 0$. We denote the interest rate with $r > 0$, the appreciation rate of the stock prices with $\mu \in \mathbb{R}, \mu > r$ and the volatility of the stock prices with $\sigma > 0$.

8.1. The Model

The cumulative claim process C_t can be modelled by a Brownian motion with drift. The integral $\int_0^t bdB_s$ can be interpreted as the fluctuations around the main claims,

$$dC_t = adt - bd\tilde{B}_t. \tag{58}$$

The insurer collects premium continuously at the constant rate $c_0 = (1+\theta)a$, where $\theta > 0$ denotes the safety loading. The dynamics of the surplus process $\{R_t\}_{t\in[0,T]}$ are given by $dR_t = c_0 dt - dC_t$. Thus, it can be followed that

$$dR_t = a\theta dt + bd\tilde{B}_t \tag{59}$$

with

$$R_0 = u > 0, \tag{60}$$

where u denotes the insurer's initial reserve. The surplus process subject to this choice is denoted by X_t, and the asset dynamics are given as:

$$\begin{aligned} dX_t = {} & (rX_t + \pi_t(\mu - r)X_t + (\theta - \eta q_t)a)dt + \sigma\pi_t X_t dW_t + b(1-q_t)d\tilde{B}_t + \\ & + \pi_t \int_{\mathbb{R}_0} g(X_{t^-}, \theta_{t^-}, u)N(dt^-, du), \\ & X_0 = x > 0. \end{aligned} \tag{61}$$

We consider the investment-reinsurance strategy $\phi = (\pi, q)$ as a control process designed to capture the optimal results of the firm.

8.1.1. Insider information

The investor possesses inside information that is introduced in form of filtration as in the models above. Let $\mathbb{F}$ be a general filtration. We denote by $A^{\mathbb{F}}$ the class of admissable strategies $\phi = (\pi_t, q_t)$ that satisfy the conditions:

1) π_t and q_t are $\mathbb{F}$ progressively measurable,
2) $0 \le \pi_t \le 1$ and $0 \le q_t \le 1$,
3) $\mathbb{E}\left(\int_0^T (\pi_t)^2 dt\right) < \infty$.

8.2. The Optimization Problem

Under the mentioned framework optimal life insurance and consumption can be thought as an area of only point: The objective is to maximize the expected utility from terminal wealth for the insurer taking into account the extra information B_{T_0} that she possesses; thus the final problem is given as:

$$V(t,x,m) = \sup_{\phi \in A^G} \mathbb{E}[U(X_T)|(X_t, M_t) = (x,m)], \tag{62}$$

subject to

$$dX_t = (rX_t + \pi_t\,(\mu - r)X_t + (\theta - \eta q_t)a + (1-q_t)(\delta_0 - M_t)b)dt + \sigma\pi_t\, X_t dW_t + \\ + b(1-q_t)d\tilde{B}_t + \pi_t \int_{\mathbb{R}_0} g(X_{t^-}, \theta_{t^-}, u)N(dt^-, du), \tag{63}$$

$$dM_t = \frac{1}{T_0 - t}d\tilde{B}_t, \tag{64}$$

where $X(t)$ is the wealth of the insurer and $M(t)$ is the stochastic part of the information drift $\delta(t)$. The Hamilton-Jacobi-Bellman generator for this process is given as:

$$\mathfrak{B}f(t,x,m) = f_t + (rx + \pi(\mu - r)x + (\theta - \eta q)a)f_x + (1-q)(\delta_0 - m)bf_x + \\ + \frac{1}{2}\sigma^2\pi^2 f_{xx} + \frac{1}{2}b^2(1-q)^2 f_{xx} + \frac{1}{2}\left(\frac{1}{T_0 - 1}\right)^2 f_{mm} + \frac{b(1-q)}{T_0 - t}f_{mx} + \pi_t \int_{\mathbb{R}_0} f(\theta(X_{t^-}, \theta_{t^-}) - \\ - f(X_{t^-}, \theta_{t^-}))p(dt^-, du), \tag{65}$$

where $f(t,x,m) \in C^{1,2,2}$.

The optimal policy for the insurer with the expected utility of its terminal wealth and investment part in the risky asset is

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2 x}\frac{V_x}{V_{xx}}, \tag{66}$$

and the reinsurance part of total claims is equal to:

$$q^*(t,x,m) = 1 + \left(\frac{a\eta}{b} + (\delta_0 - m)\right)\frac{V_x}{bV_{xx}} + \frac{1}{b(T_0 - t)}\frac{V_{mx}}{V_{xx}}, \tag{67}$$

where $V(t,x,m)$ is the optimal solution to the Cauchy problem:

$$V_t + \big(rx + a(\theta - \eta)\big)V_x - \frac{1}{2}\left[\left(\frac{\mu - r}{\sigma}\right)^2 + \left(\frac{a\eta}{b}\right)^2 + (\delta_0 - m)^2\right]\frac{V_x^2}{V_{xx}} - \left(\frac{a\eta}{b}\right)(\delta_0 - m)\frac{V_x^2}{V_{xx}} - \\ \left(\frac{1}{T_0 - t}\right)\left[\left(\frac{a\eta}{b}\right)^2 + (\delta_0 - m)\right]\frac{V_x V_{mx}}{V_{xx}} - \frac{1}{2}\left(\frac{1}{T_0 - t}\right)^2\frac{V_x^2}{V_{xx}} + \frac{1}{2}\left(\frac{1}{T_0 - t}\right)^2 V_{mm} = 0. \tag{68}$$

For the proof of the Cauchy problem we refer to Yannacopoulos [32]. The Hamilton-Jacobi-Bellman (HJB) equation for the insurer which is generated for the state process subject to the choice π_t and q_t takes the following form:

$$\mathfrak{B}f(t,x,m) = f_t + (rx + \pi(\mu - r)x + (\theta - \eta q)a)f_x + (1-q)(\delta_0 - m)bf_x$$
$$+\frac{1}{2}\sigma^2\pi^2x^2f_{xx} + \frac{1}{2}b^2(1-q)^2f_{xx} + \frac{1}{2}\left(\frac{1}{T_0-t}\right)^2 f_{mm} + \frac{b(1-q)}{T_0-1}f_{mx}, \quad (72)$$

where $f(t,x,m) \in C^{1,2,2}$.

8.2.1. Optimal Strategies

The resulting optimal strategies are obtained as solutions for the PDE. The optimality condition for investing in the risky asset is:

$$\pi^*(t,x) = -\frac{\mu-r}{\sigma^2 x}\frac{V_x}{V_{xx}}. \quad (73)$$

The reinsurance proportion of the total claims is equal to:

$$q^*(t,x,m) = 1 + \left(\frac{a\eta}{b} + (\delta_0 - m)\right)\frac{V_x}{bV_{xx}} + \frac{1}{b(T_0-t)}\frac{V_{mx}}{V_{xx}}. \quad (74)$$

By solving the HJB equations, closed-form solutions for the optimal investment and reinsurance policies for both firms are found, taking into account their information sets, in order to maximize their expected utility from terminal wealth.

9. CONCLUSION

Regime switching dynamics and abrupt fluctuations which are a usual part of financial markets are able to be captured in Stochastic Hybrid System specifications with a Brownian motion part and additionally a jump part. These kinds of models allow for more realistic investigation of portfolio optimization and utility maximization in financial markets. The models are able to account for the efficient hypothesis methodology. Insider information is incorporated in models, where filtration gives the necessary tool to develop optimal consumption and investment rates. Various SHS model applications provide specific results in financial conceptions, where different specifications in HJB functions generate the solutions.

In Okur and Davis [23] forward integrals are used to determine the optimal investment and consumption rates in markets which are driven by Brownian motion and in markets with an additional jump component. The proposed model extends the results of Di Nunno et al. [8] by considering both, the optimal portfolio and consumption rate choices of an insider, when her portfolio is allowed to anticipate the future optimal wealth process.

In the insurance and reinsurance model developed by Yannacopoulos et al. [32] the optimization sequences have a different impulse. In this model, an insurer and a reinsurer can both invest their surplus in the same financial market. Furthermore, the insurer has the possibility to pass to the reinsurer part of the underlying risk involved with its claim process, by purchasing proportional reinsurance coverage. The introduction of inside information is accomplished by letting the insurer to have some explicit information about the realization of

its claim process, which is not available to the reinsurer. In this sense, it is shown that the insurer always determines its optimal reinsurance strategy according to the extra information available at its disposal, whereas the optimal investment decision is always irrelevant of this knowledge. Yannacopoulos et al. [32] prove that as the level of the extra information increases, there is less proportional reinsurance demand, whereas as the level of the extra information decreases, the insurer purchases more proportional reinsurance coverage. Significant outliers in the financial and economic data have a substantial effect on the models' composition. Thus, a new model is needed to have the correct specification. In this sense, these dynamics can be seen as a part of a stochastic control problem which is to be formed according to the new conditions. Through further variations of SHS models, a generalization in optimization of consumption and investment decisions in markets with excessive volatilities can be achieved. Dynamics of the SHS models' jump parts and the discrete component give the possibility to account for the modelling of outliers and excessive volatility (skewness) from the financial data.

A main field of potential applications in finance should be the modelling of sudden changes in information that prevail in highly dynamic stock markets. Expectations will change, and thus, the price building process is expected to involve the emergent information. The above mentioned SHS properties are able to account for a new format of asset pricing and risk management mechanisms. The pricing of the stock returns and the pricing of options will gain new perspectives through implementations of SHS frameworks.

Change in dynamics of macroeconomic variables can be conceived as applications of Stochastic Hybrid Systems in economics. A parallelism can be drawn to Kydland and Prescott's work [19]. They trace an introduction of technology shocks in a general equilibrium of a dynamic and stochastic macroeconomic model based on microeconomic foundations. This requires solving a set of interrelated dynamic optimization problems, where consumers and firms make decisions based on current and expected future paths for prices and policy variables, and where the equilibrium price sequences are such that private-sector decisions are consistent with clearing markets at all points in time and all states of the world. Kydland and Prescott [19] showed that this analysis could be carried out in practice by extensive use of numerical methods. Their empirical approach relied on model simulation, based on calibration, and on comparing the synthetic data from simulations with actual data. Analogously to these considerations, numerical implementations of specified SHS models are an area for future research in economics and finance. In extension, benefiting from the flexible solution methodology of HJB functions, approximations for non-linear Hamilton-Jacobi PDEs with boundary conditions can be applied as solutions to Stochastic Hybrid models.

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